

Davenport constant with weights

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Abstract

For the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ and any non-empty $A \in \mathbb{Z}$. We define the Davenport constant of G with weight A , denoted by $D_A(n)$, to be the least natural number k such that for any sequence (x_1, \dots, x_k) with $x_i \in G$, there exists a non-empty subsequence $(x_{j_1}, \dots, x_{j_l})$ and $a_1, \dots, a_l \in A$ such that $\sum_{i=1}^l a_i x_{j_i} = 0$. Similarly, we define the constant $E_A(n)$ to be the least $t \in \mathbb{N}$ such that for all sequences (x_1, \dots, x_t) with $x_i \in G$, there exist indices $j_1, \dots, j_n \in \mathbb{N}, 1 \leq j_1 < \dots < j_n \leq t$, and $\vartheta_1, \dots, \vartheta_n \in A$ with $\sum_{i=1}^n \vartheta_i x_{j_i} = 0$. In the present paper, we show that $E_A(n) = D_A(n) + n - 1$. This solve the problem raised by Adhikari and Rath [3], Adhikari and Chen [2], Thangadurai [12] and Griffiths [10].

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1 Introduction

For an abelian group G , the Davenport constant $D(G)$ is defined to be the smallest natural number k such that any sequence of k elements in G has a non-empty subsequence whose sum is zero (the identity element). Another interesting constant $E(G)$ is defined to be the smallest natural number k such that any sequence of k elements in G has a subsequence of length $|G|$ whose sum is zero.

The following result due to Gao [7] connects these two invariants.

Theorem 1.1 *If G is a finite abelian group of order n , then $E(G) = D(G) + n - 1$.*

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For a finite abelian group G and any non-empty $A \in \mathbb{Z}$, Adhikari and Chen [2] defined the Davenport constant of G with weight A , denoted by $D_A(G)$, to be the least natural number k such that for any sequence (x_1, \dots, x_k) with $x_i \in G$, there exists a non-empty subsequence $(x_{j_1}, \dots, x_{j_l})$ and $a_1, \dots, a_l \in A$ such that $\sum_{i=1}^l a_i x_{j_i} = 0$. Clearly, if G is of order n , it is equivalent to consider A to be a non-empty subset of $\{0, 1, \dots, n-1\}$ and cases with $0 \in A$ are trivial.

Similarly, for any such set A , for a finite abelian group G of order n , the constant $E_A(G)$ is defined to be the least $t \in \mathbb{N}$ such that for all sequences (x_1, \dots, x_t) with $x_i \in G$, there exist indices $j_1, \dots, j_n \in \mathbb{N}, 1 \leq j_1 < \dots < j_n \leq t$, and $\vartheta_1, \dots, \vartheta_n \in A$ with $\sum_{i=1}^n \vartheta_i x_{j_i} = 0$.

For the group $G = \mathbb{Z}/n\mathbb{Z}$, we write $E_A(n)$ and $D_A(n)$ respectively for $E_A(G)$ and $D_A(G)$. In the cases $A = \{1\}$, $\{-1, 1\}$, \mathbb{Z}_n^* or $A = (a_1, \dots, a_r)$ with $\gcd(a_2 - a_1, \dots, a_r - a_1, n) = 1$ or $n = p$ is a prime, it is proved that $E_A(n) = D_A(n) + n - 1$. The following conjecture has been raised by Adhikari and Rath [3], Adhikari and Chen [2], Thangadurai [12] and Griffiths [10], they seems to believe that Conjecture 1.1 is true and have proved it in some special cases.

Conjecture 1.1 *For any non-empty set $A \in \mathbb{Z}$, $E_A(n) = D_A(n) + n - 1$.*

The main purpose of the present paper is to prove Conjecture 1.1. By using the main theorem of Devos, Goddyn and Mohar [5] and a recently proved theorem of the authors [13], we shall prove the following theorem

Theorem 1.2 *For any non-empty set $A \in \mathbb{Z}$, $E_A(n) = D_A(n) + n - 1$.*

Throughout this paper, let G be an additive finite abelian group. $\mathcal{F}(G)$ denotes the free abelian monoid with basis G , the elements of which are called *sequences* (in G). A sequence of not necessarily distinct elements from G will be written in the form $S = g_1 \cdots g_k = \prod_{i=1}^k g_i = \prod_{g \in G} g^{\mathbf{v}_g(S)} \in \mathcal{F}(G)$, where $\mathbf{v}_g(S) \geq 0$ is called the *multiplicity* of g in S . We call $|S| = k$ the *length* of S , $\mathbf{h}(S) = \max\{\mathbf{v}_g(S) | g \in G\} \in [0, |S|]$ the maximum of the multiplicities of S , $\text{supp}(S) = \{g \in G : \mathbf{v}_g(S) > 0\}$ the *support* of S . For every $g \in G$ we set $g + S = (g + g_1) \cdots (g + g_k)$.

We say that S contains some $g \in G$ if $\mathbf{v}_g(S) \geq 1$ and a sequence $T \in \mathcal{F}(G)$ is a *subsequence* of S if $\mathbf{v}_g(T) \leq \mathbf{v}_g(S)$ for every $g \in G$, denoted by $T|S$. Furthermore, by $\sigma(S)$ we denote the sum of S , (i.e. $\sigma(S) = \sum_{i=1}^k g_i = \sum_{g \in G} \mathbf{v}_g(S)g \in G$). For every $k \in \{1, 2, \dots, |S|\}$, let $\sum_k(S) = \{g_{i_1} + \dots + g_{i_k} | 1 \leq i_1 < \dots < i_k \leq |S|\}$, $\sum_{\leq k}(S) = \cup_{i=1}^k \sum_i(S)$, and let $\sum(S) = \sum_{\leq |S|}(S)$.

Let S be a sequence in G . We call S a *zero-sum sequence* if $\sigma(S) = 0$.

Also, we follow the same terminologies and notations as in the survey article [8] or in the book [9].

2 Lemmas

First, we need a result on the sum of l finite subsets of G . If $\mathbf{A} = (A_1, A_2, \dots, A_m)$ is a sequence of finite subsets of G , and $l \leq m$, we define

$$\sum_l(\mathbf{A}) = \{a_{i_1} + \dots + a_{i_l} : 1 \leq i_1 < \dots < i_l \leq m \text{ and } a_{i_j} \in A_{i_j} \text{ for every } 1 \leq j \leq l\}.$$

So $\sum_l(\mathbf{A})$ is the set of all elements which can be represented as a sum of l terms from distinct members of \mathbf{A} . The following is the main result of Devos, Goddyn and Mohar [5].

Theorem DGM *Let $\mathbf{A} = (A_1, A_2, \dots, A_m)$ be a sequence of finite subsets of G , let $l \leq m$, and let $H = \text{stab}(\sum_l(\mathbf{A}))$. If $\sum_l(\mathbf{A})$ is nonempty, then*

$$|\sum_l(\mathbf{A})| \geq |H|(1 - l + \sum_{Q \in G/H} \min\{l, |\{i \in \{1, \dots, m\} : A_i \cap Q \neq \emptyset\}|\}).$$

We still need the following new result on Davenport's constant [13].

Theorem YZ *Let G be a finite abelian group of order n and Davenport constant $D(G)$. Let $S = 0^{h(S)} \prod_{g \in G} g^{v_g(S)} \in \mathcal{F}(G)$ be a sequence with a maximal multiplicity $h(S)$ attained by 0 and $|S| = t \geq n + D(G) - 1$. Then there exists a subsequence S_1 of S with length $|S_1| \geq t + 1 - D(G)$ and $0 \in \sum_k(S_1)$ for every $1 \leq k \leq |S_1|$. In particular, for every sequence S in G with length $|S| \geq n + D(G) - 1$, we have*

$$0 \in \sum_{km}(S), \text{ for every } 1 \leq k \leq (|S| + 1 - D(G))/m,$$

where m is the exponent of G .

3 Proof of Theorem 1.2

Proof. The proof of $E_A(n) \geq D_A(n) + n - 1$ is easy, so it is sufficient to prove the reverse inequality.

For any non-empty set $A = \{a_1, \dots, a_r\} \subset \mathbb{Z}$ and a cyclic group $G = \mathbb{Z}/n\mathbb{Z}$, let $t = D_A(n) + n - 1$ and $S = x_1 \dots x_t$ is any sequence in G with length $|S| = t = D_A(n) + n - 1$. Put

$$A_i = Ax_i = \{a_1x_i, \dots, a_rx_i\} \text{ for } i = 1, \dots, t$$

and $\mathbf{A} = (A_1, \dots, A_t)$. It suffices to prove that $0 \in \sum_n(\mathbf{A})$.

We shall assume (for a contradiction) that the theorem is false and choose a counterexample (A, G, S) so that $n = |G|$ is minimum, where G is a cyclic group of order n , A is a finite subset of \mathbb{Z} and $S = x_1 \dots x_t$ is a sequence in G such that

$$0 \notin \sum_n(\mathbf{A}).$$

Next we will show that our assumptions imply $H = \text{stab}(\sum_n(\mathbf{A})) = \{0\}$. Suppose (for a contradiction) that $H = \text{stab}(\sum_n(\mathbf{A})) \neq \{0\}$ and let $\varphi : G \longrightarrow G/H$ denote the canonical homomorphism and $\varphi(x_i)$ the image of x_i for $1 \leq i \leq t$. Let $\mathbf{A}_\varphi = (\varphi(A_1), \dots, \varphi(A_t))$. By our assumption for the minimal of $|G|$, the theorem holds for $(A, \varphi(G), \varphi(S))$. Since $n > |\varphi(G)| + D_A(\varphi(G)) - 1$, $|\varphi(G)| \mid n$ and $D_A(G) \geq D_A(\varphi(G))$, repeated applying the theorem to the sequence $\varphi(S) = \varphi(x_1), \dots, \varphi(x_t)$ we have

$$\varphi(0) = \varphi(H) \in \sum_n(\mathbf{A}_\varphi),$$

thus $0 \in H \subset \sum_n(\mathbf{A})$. This contradiction implies that $H = \text{stab}(\sum_n(\mathbf{A})) = \{0\}$.

If there is an element $a \in G$ such that $|\{j \in \{1, \dots, t\} : a \in A_j\}| \geq n$, then $0 \in \sum_n(\mathbf{A})$, a contradiction. Therefore we may assume that for every $a \in G$, $|\{j \in \{1, \dots, t\} : a \in A_j\}| \leq n$. Let r be the number of $i \in \{1, \dots, t\}$ with $|A_i| = 1$, by Theorem DGM and the assumptions, we have

$$\begin{aligned} n - 1 &\geq \sum_n(\mathbf{A}) \geq 1 - n + \sum_{a \in G} \min\{n, |\{j \in \{1, \dots, t\} : a \in A_j\}|\} \\ &= 1 - n + \sum_{i=1}^t |A_i| \geq 1 - n + 2(n + D_A(G) - 1 - r) + r. \end{aligned}$$

It follows that

$$r \geq 2D_A(G). \quad (1)$$

Without loss of generality, we may assume that x_1, \dots, x_r are all the elements in $\{x_1, \dots, x_t\}$ such that $|A_i| = 1$, and x_1 is the element in $\{x_1, \dots, x_r\}$ such that $a_1 x_1$ attains the maximal multiplicity in the sequence $S_1 = (a_1 x_1) \cdots (a_1 x_r)$. Observe that $\sum_n(\mathbf{A}) = \sum_n(A(x_1 - x_u), \dots, A(x_t - x_u))$ for every $1 \leq u \leq r$. Therefore without loss of generality we may assume that $a_1 x_1 = 0$ and $v_0(S_1) = h(S_1)$ for the sequence

$$S_1 = (a_1 x_1) \cdots (a_1 x_r) = 0^{h(S_1)} (a_1 x_{h(S_1)+1}) \cdots (a_1 x_r). \quad (2)$$

Let $H_1 = \langle x_1, \dots, x_r \rangle$ be the group generated by x_1, \dots, x_r , $H = a_1 H_1$. We have the following claim.

Claim: $D_A(G) \geq D_A(H_1) \geq D(H) = |H|$.

The last equality of the Claim follows from the fact that H is a subgroup of the cyclic group G . The first inequality in the Claim is obvious, so we only need to prove that $D_A(H_1) \geq D(H)$. Suppose that $W = y_1 \cdots y_{D(H)-1}$ is a zero-sum free sequence in H . Since $H = a_1 H_1$, we have $y_i = a_1 w_i, w_i \in H_1, i = 1, \dots, r$. Further, it is easy to see that $Aw_i = a_1 w_i, i = 1, \dots, r$ by the definition of H_1 , so $w_1 \cdots w_{D(H)-1}$ is a zero-sum free sequence in H_1 with respect to the weight A , thus $D_A(H_1) \geq D(H)$ and the Claim follows.

By the Claim, (1), (2) and Theorem YZ, S_1 has a subsequence S_2 of length $|S_2| = s \geq r + 1 - |H|$ such that $0 \in \sum_l(S_2)$ for every $1 \leq l \leq s$. Without loss of generality, we may assume that $S_2 = (a_1x_1) \cdots (a_1x_s)$.

If $s \geq n$ then $0 \in \sum_n(S_2) \subset \sum_n(\mathbf{A})$, we are done.

If $s < n$, then $|x_{s+1} \cdots x_t| = t - s = n - 1 + D_A(G) - s \geq D_A(G)$. Repeated using the definition of $D_A(G)$, there exists an integer v such that $v \leq n$, $t - s - v \leq D_A(G) - 1$ and

$$0 \in \sum_v((A_{s+1}, \cdots A_t)).$$

Since $\sum_l(S_2) = \sum_l((A_1, \cdots, A_s))$ and $0 \in \sum_l((A_1, \cdots, A_s))$ for every $1 \leq l \leq s$, we have

$$0 \in \sum_{v+k}(\mathbf{A}) \quad \text{for every } 0 \leq k \leq s.$$

Therefore $0 \in \sum_n(\mathbf{A})$ since $v + s \geq t + 1 - G_A(G) \geq n$. This completes the proof of the theorem. □

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